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Symmetric central configurations of the spatial n-body problem

FERRAN CEDÓ, JAUME LLIBRE

Departement de Mathemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
Spain

Abstract. *We characterize the non-planar central configurations of the spatial n-body problem with equal masses which are orbits of a finite group of isometries of $\mathbf{R}^3$. As a corollary we obtain that the spatial n-body problem with equal masses and $n > 5$ has at least two equivalence classes of non-planar central configurations modulo homotheties and rotations.*

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $q_1, \ldots, q_n \in \mathbf{R}^d$ denote the positions of n bodies with masses $m_1, \ldots, m_n$ respectively. Their motion is described by the equations

$$m_i \ddot{q}_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} m_i m_j \frac{q_i - q_j}{|q_i - q_j|^3} = -\nabla_i V(q) \qquad \text{for} \quad i = 1, \ldots, n; \tag{1.1}$$

where

$$V(q) = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$$

is the potential energy $V : \mathbf{R}^{dn} \setminus \Delta \to \mathbf{R}$ and $\Delta = \bigcup_{i<j} \Delta_{ij}$ is the set of collisions because $\Delta_{ij} = \{(q_1, \ldots, q_n) \in \mathbf{R}^{dn} \mid q_i = q_j\}$. We fix the center of mass $\sum m_i q_i$

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at the origin of $\mathbf{R}^d$. Thus the *configuration space* for the n-body problem is $X \setminus \Delta$, where $X = \{(q_1, \dots, q_n) \in \mathbf{R}^{dn} \mid \sum m_i q_i = 0\}$.

A point $q = (q_1, \dots, q_n) \in X \setminus \Delta$ is a *central configuration* of the n-body problem if there exists a scalar-valued function $\phi(t)$ such that a solution of (1.1) has the form $\phi(t) q$.

Of course, if q is a central configuration then so is aq for any non-zero real number a. Therefore in studying central configurations we may restrict our attention to the ellipsoid $E = \{q \in X \mid \sum m_i |q_i|^2 = 1\}$. Now suppose that $q \in E$ is a central configuration. From (1.1) and the homogeneity of $V(q)$, we have

$$\ddot{\phi} m_i q_i = -\phi^{-2} \nabla_i V(q) \qquad \text{for} \quad i = 1, \dots, n; \tag{1.2}$$

that is,

$$\lambda m_i q_i = \nabla_i V(q), \qquad \lambda = -\ddot{\phi}\phi^2 = -V(q).$$

Conversely, if we assume that $\lambda m_i q_i = \nabla_i V(q)$ for $i = 1, \dots, n$ and $q \in E$, then a solution $\phi(t)$ of the equation $\ddot{\phi} = \phi^{-2} V(q)$ provides a solution of (1.2); so q is a central configuration.

The group $SO(d)$ acts diagonally on E leaving invariant the set of solutions of (1.1). Then, if C_d is the set of central configurations of E for the n-body problem in $\mathbf{R}^d$, let $\bar{C}_d$ be the set of equivalence classes of C_d, where $q_1, q_2 \in C_d$ are equivalent if they differ by a rotation of $SO(d)$. An element of $\bar{C}_d$ is denoted by $\bar{q}$ and it is a central configuration modulo homotheties and rotations.

The following facts are known (see for instance the references of [4] and [6]).

(1) If $n \geq 2$, $\bar{C}_1$ has exactly $n!/2$ elements (Moulton).

(2) If $n = 2$, $\bar{C}_d$ has one element for $d = 2, 3$.

(3) If $n = 3$, $\bar{C}_d$ has five elements for $d = 2, 3$ (Euler and Lagrange).

(4) If $n \geq 4$ and $d = 2$ there are several results due mainly to Palmore, see also [5].

(5) If $n \geq 4$, $d = 3$ and $m_1 = \dots = m_n$, then, by using equivariant Morse theory, Pacella shows that $\bar{C}_d$ has at least one non-planar element.

The question as to whether $\bar{C}_d$, for $d = 2, 3$, is finite for every choice of $m_1, \dots, m_n$ is an open problem [6].

From now on *we shall consider the n-body problem in* $\mathbf{R}^3$ *with equal masses*, i.e. $m_1 = \dots = m_n = m$. A point $q = (q_1, \dots, q_n) \in X \setminus \Delta$ will be called a *symmetric central configuration* if it is a central configuration of the n-body problem in $\mathbf{R}^3$ with equal masses and the set $\{q_1, \dots, q_n\}$ is an orbit by the action of a finite subgroup of $O(3)$ on $\mathbf{R}^3$, that is, by the action of a finite group of isometries (fixing the origin) of $\mathbf{R}^3$. The next lemma facilitates the study of the symmetric central configurations.

Table I. The points of the symmetric central configurations are the vertices of the polyhedra. We refer the reader to [2] for the notation on the finite groups of isometries.

n	*Classes of symmetric central configurations*	*Orbits by the action of the groups*
4	Regular tetrahedron	$[2,2]^+, [2^+,4^+], [2^+,4]$, $[3,3]^+, [3,3]$.
6	Regular octahedron (antiprism)	$[2,3]^+, [2^+,6^+], [2^+,6]$, $[3,3]^+, [3,3]$. $[3,4]^+, [3^+,4], [3,4]$.
8	Cube (prism)	$[2,2], [2^+,4], [2,4^+]$, $[2,4]^+, [2,4]$. $[3,4]^+, [3^+,4], [3,4]$.
12	Truncated regular tetrahedron	$[3,3]^+, [3,3]$.
12	Cuboctahedron	$[3,3]^+, [3,3]$, $[3,4]^+, [3^+,4], [3,4]$.
12	Regular icosahedron	$[3,3]^+, [3^+,4]$, $[3,5]^+, [3,5]$.
20	Regular dodecahedron	$[3,5]^+, [3,5]$.
24	Truncated regular octahedron	$[3,3], [3,4]^+, [3,4]$.
24	Truncated cube; Parallelly bevelled cube	$[3,4]^+, [3^+,4], [3,4]$.
24	Laevo snub cube; Dextro snub cube	$[3,4]^+$.
30	Icosidodecahedron	$[3,5]^+, [3,5]$.
48	Parallelly bevelled truncated cube	$[3,4]$.
60	Truncated regular icosahedron; Truncated regular dodecahedron; Parallelly bevelled regular icosahedron	$[3,5]^+, [3,5]$.
60	Laevo snub dodecahedron; Dextro snub dodecahedron	$[3,5]^+$.
120	Parallelly bevelled truncated regular icosaedron	$[3,5]$.
$n = 2q > 6$	Antiprism	$[2,q]^+, [2^+,2q^+], [2^+,2q]$.
$n = 2q > 4$, $4 \nmid n$	Prism with regular bases	$[2,q]^+, [2,q^+], [2,q]$.
$n = 4q > 4$	Prism with regular bases	$[2,q], [2^+,2q]$, $[2,2q^+], [2,2q]^+, [2,2q]$.

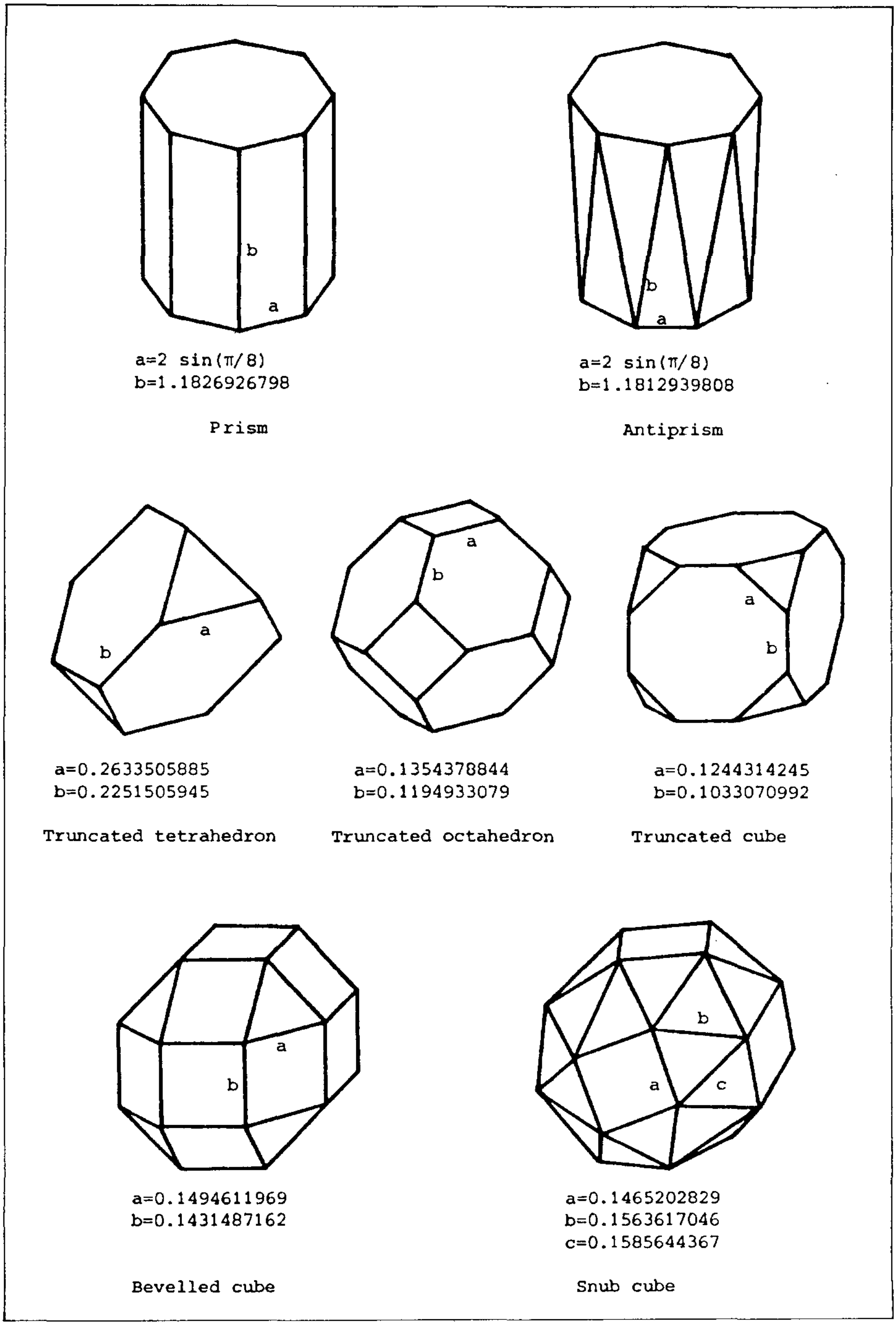


Figure 1. Non-regular symmetric central configurations.

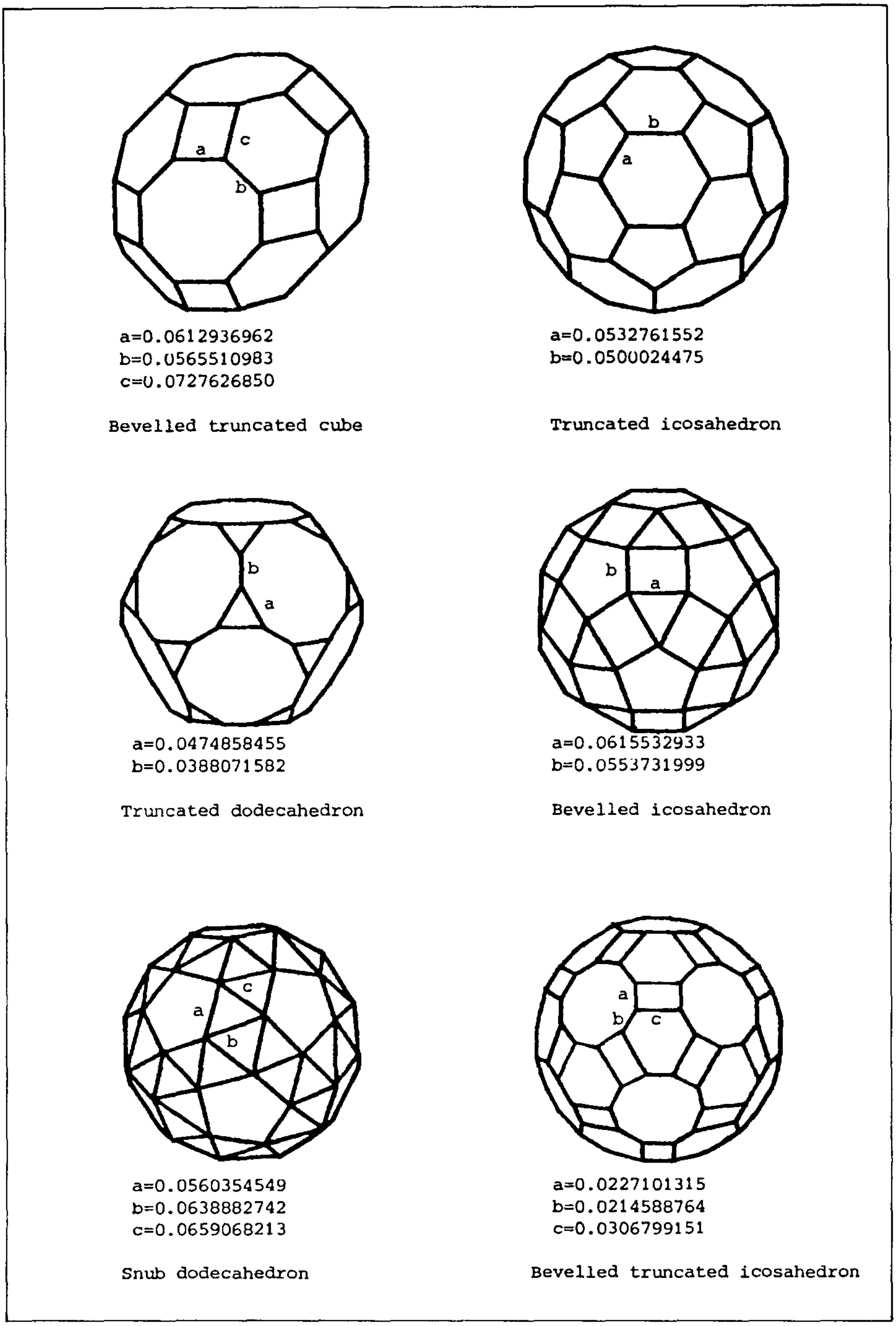


Figure 1. (cont.).

LEMMA 1.1. *Let* $\{q_1, \ldots, q_n\} \subset \mathbf{R}^3 \setminus \{0\}$ *be an orbit by the action of a finite subgroup* G *of* $O(3)$. *Then* $(q_1, \ldots, q_n)$ *is a symmetric central configuration if and only if*

$$\mu q_1 = \sum_{j=2}^{n} \frac{q_1 - q_j}{|q_1 - q_j|^3} \qquad \text{with} \quad \mu > 0. \tag{1.3}$$

Proof. From (1.2), we have that $(q_1, \ldots, q_n)$ is a central configuration if and only if

$$\mu q_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_i - q_j}{|q_i - q_j|^3} \qquad \text{with} \quad \mu = -\frac{\ddot{\phi}\phi^2}{m} = -\frac{V(q)}{m^2 \sum_{i=1}^{n} |q_i|^2} > 0, \tag{1.4}$$

for $i = 1, \ldots, n$. So the "only if" part follows. Since $\{q_1, \ldots, q_n\}$ is an orbit by the action of G, there exists $\varphi \in G$ such that $\varphi(q_1) = q_i$. Therefore (1.4) follows from (1.3), and the "if" part is proved. ∎

Our main result is the following one:

THEOREM A. *The non-planar symmetric central configurations modulo homotheties and rotations are given in Table I and Figure 1.*

The existence of the symmetric central configurations given in Table I is proved analytically, but the uniqueness is shown numerically, see Sections 3,4,5 and 6.

Remark 1.2. If $(q_1, \ldots, q_n)$ is a symmetric central configuration for the n-body problem, then $(q_1, \ldots, q_n, 0)$ is a central configuration for the $(n+1)$-body problem.

By Theorem A and Remark 1.2, the following result follows immediately.

COROLLARY B. $\bar{C}_3$ *has at least* 2 *non-planar elements for the* n*-body problem with equal masses when* $n > 5$.

This corollary improves the result (5) of Pacella, see an analytic proof of it in Section 3.

2. ORBITS BY THE ACTION OF A FINITE GROUP OF ISOMETRIES

Now, we summarize the well-known results on finite groups of isometries.

A finite subgroup F of $O(3)$ generated by reflections in any number of planes is equally well generated by reflections in all their transforms: a configuration of planes that is symmetrical by reflections in each one. These planes all pass through the origin O and determine a corresponding configuration of great circles on S^2. These great circles decompose S^2 into a finite number of regions (namely hemispheres, lunes, or spherical triangles) whose angles are submultiples of π [1]. All the regions are congruent (with a possible reversal of sense), since each reflects into its neighbours. These regions are called *fundamental regions.* There is a point inside or on the boundary of one of these regions for each orbit by the action of F on S^2.

Any finite subgroup G of $O(3)$ is a subgroup of a finite subgroup F of $O(3)$ generated by reflections [2].

The *finite groups generated by reflections* are:

$[1]$, of order 2, generated by the reflection in a single plane which cuts S^2 into two hemispheres.

$[q]$, of order $2q$, generated by two reflections R_1, R_2 in two planes which cut themselves with angle π/q $(q \geq 2)$.

$[2, q]$, of order $4q$, with a fundamental region given by a spherical triangle of angles $\pi/2$, π/q, $\pi/2$ $(q \geq 2)$.

$[3, 3]$, of order 24, with a fundamental region given by a spherical triangle of angles $\pi/3$, $\pi/3$, $\pi/2$.

$[3, 4]$, of order 48, with a fundamental region given by a spherical triangle of angles $\pi/3$, $\pi/4$, $\pi/2$.

$[3, 5]$, of order 120, with a fundamental region given by a spherical triangle of angles $\pi/3$, $\pi/5$, $\pi/2$.

The groups $[p, q]$ are generated by reflections R_1, R_2, R_3 satisfying

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^2 = E,$$

where E is the identity.

The *proper non-trivial subgroups of the finite reflection groups* are:

$[q]^+$, the subgroup of index 2 of $[q]$, generated by the rotation R_1R_2 $(q \geq 2)$.

$[p, q]^+$, the subgroup of index 2 of $[p, q]$, generated by the rotations R_1R_2, R_2R_3.

$[p^+, q]$ (q even), the subgroup of index 2 of $[p, q]$, generated by R_1R_2 and R_3.

$[p, q^+]$ (p even), the subgroup of index 2 of $[p, q]$, generated by R_2R_3 and R_1.

$[2^+, q^+]$ (q even), the subgroup of index 4 of $[2, q]$, generated by $R_1R_2R_3$.

Now we start the description of the orbits by the action of the finite group of isometries on S^2.

The orbits of any point of S^2 by the action of any finite subgroup of $[q]$ are planar.

Let T' be the fundamental region T of the group $[2, q]$ minus the north pole and the equator (see Figure 2). Since the orbits of the north pole or of any point of the equator by the action of any subgroup of $[2, q]$ are planar, in what follows we only study the orbits of points in T'.

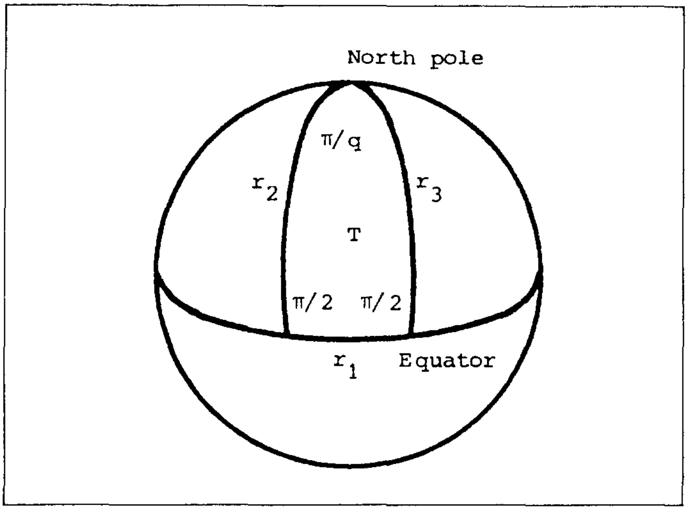


Figure 2. Fundamental region T of the group $[2,q]$. The edges r_i correspond to the planes of the reflections R_i for $i = 1,2,3$.

The orbits of any point inside T' by the action of $[2,q]$ are the vertices of *prisms* whose faces consist of two $2q$-gons (regular or not) connected by $2q$ rectangles. For the remainder points of T' the orbits by the action of $[2,q]$ are the vertices of prisms with the bases formed by two regular q-gons.

The orbits of any point inside T' by the action of $[2,q]^+$ are the vertices of *antiprisms* whose faces consist of two regular q-gons connected by $2q$ triangles. For the rest of the points of T' the orbits by the action of $[2,q]^+$ are the vertices of prisms with two regular q-gons as bases.

The orbits of any point of T' by the action of $[2,q^+]$ are the vertices of prisms with two regular q-gons as bases.

The orbits of any point inside T' by the action of $[2^+,2q]$ are the vertices of *pseudo-prisms* whose faces consist of two $2q$-gons connected by $2q$ trapezia. For the points of T' on the edge r_3 (see Figure 2), the orbits are the vertices of antiprisms with two regular q-gons as bases. For the points of T' on the edge r_2, the orbits are the vertices of prisms with two regular $2q$-gons as bases.

The orbits of any point of T' by the action of $[2^+,2q^+]$ are the vertices of antiprisms with two regular q-gons as bases.

Let T_3 be the spherical triangle of angles $\pi/3$, $\pi/3$ and $\pi/2$ (see Figure 3(a)).

The orbits of the vertices B, C of T_3 by the action of $[3,3]^+$ and $[3,3]$ consist of the vertices of regular tetrahedra. The orbit of the vertex A of T_3 by the action of $[3,3]^+$ or $[3,3]$ consists of the vertices of a regular octahedron. The orbits of any point on the two edges r_1 and r_3 of T_3 by the action of $[3,3]^+$ or $[3,3]$ are

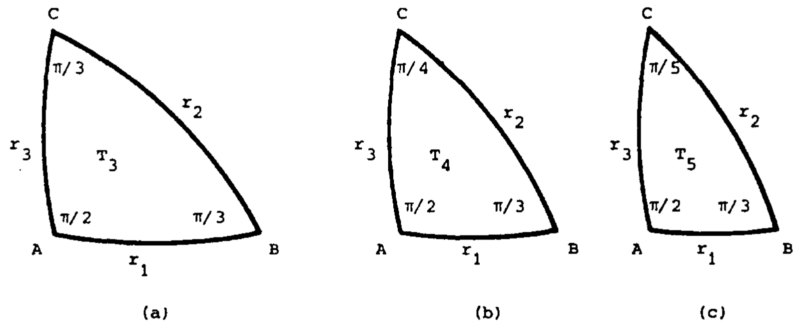


Figure 3. Fundamental regions of the groups $[3,3],[3,4],[3,5]$. The edges r_i correspond to the planes of the reflections R_i for $i=1,2,3$.

the vertices of truncated regular tetrahedra. The orbits of any point on the remainder edge of T_3 by the action of $[3,3]^+$ or $[3,3]$ are the vertices of parallelly bevelled regular tetrahedra. The orbits of any point inside T_3 by the action of $[3,3]$ consist of the vertices of parallelly bevelled truncated regular tetrahedra. The orbits of any point inside T_3 by the action of $[3,3]^+$ consist of the vertices of non-parallelly bevelled regular tetrahedra.

Let T_4 be the spherical triangle of angles $\pi/3$, $\pi/4$ and $\pi/2$ (see Figure 3(b)).

The orbit of the vertex B (resp. C or A) of T_4 by the action of $[3,4]^+$, $[3^+,4]$ or $[3,4]$ consists of the vertices of a cube (resp. regular octahedron or cuboctahedron).

The orbits of any point on the edge r_1 (resp. r_2) of T_4 by the action of $[3,4]^+$, $[3^+,4]$ or $[3,4]$ are formed by the vertices of truncated cubes (resp. parallelly bevelled cubes). The orbits of any point on the edge r_3 of T_4 by the action of $[3,4]^+$ or $[3,4]$ (resp. $[3^+,4]$) are formed by the vertices of truncated regular octahedra (resp. non-parallelly bevelled regular tetrahedra).

The orbits of any point inside T_4 by the action of $[3,4]$ consist of the vertices of parallelly bevelled truncated cubes. The orbits of any point inside T_4 by the action of $[3,4]^+$ are formed by the vertices of laevo or dextro non-parallelly bevelled cubes (or snub cubes). The orbits of any point inside T_4 by the action of $[3^+,4]$ consist of the vertices of laevo or dextro snub parallelly bevelled cubes (see Figure 4).

Let T_5 be the spherical triangle of angles $\pi/3$, $\pi/5$ and $\pi/2$ (see Figure 3(c)).

The orbit of the vertex C (resp. B or A) of T_5 by the action of $[3,5]^+$ or $[3,5]$ consists of the vertices of a regular icosahedron (resp. regular dodecahedron or icosidodecahedron).

The orbits of any point on the edge r_1 (resp. r_3 or r_2) of T_5 by the action of

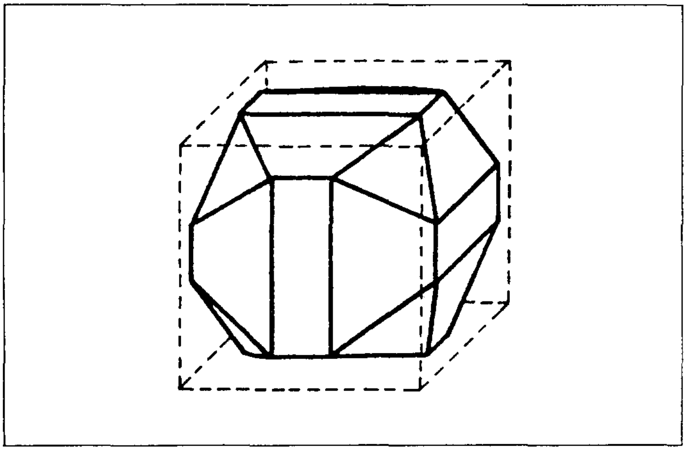


Figure 4. Snub parallelly bevelled cube.

$[3,5]^+$ or $[3,5]$ consist of the vertices of truncated regular dodecahedra (resp. truncated regular icosahedra or parallelly bevelled regular icosahedra).

The orbits of any point inside of T_5 by the action of $[3,5]$ (resp. $[3,5]^+$) consist of the vertices of parallelly bevelled truncated regular icosahedra (resp. laevo or dextro snub dodecahedra).

3. PRISMS, ANTIPRISMS AND PSEUDO-PRISMS

First we shall study the prisms and the antiprisms with regular bases.

Consider $P_i = (\cos(2\pi i/n), \sin(2\pi i/n), a)$ and $Q_i = (\cos((2\pi i/n) + \beta), \sin((2\pi i/n) + \beta), -a)$ for $i = 0, \ldots, n-1$, with $n \geq 3$, $a > 0$ and $0 \leq \beta \leq \pi/n$. If $\beta = 0$ then these points are the vertices of a prism, otherwise they are the vertices of an antiprism.

To study what of these prisms and antiprisms are symmetric central configurations we must to solve the system (1.3) for $(q_1, \ldots, q_{2n}) = (P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1})$, namely

$$\sum_{i=1}^{n-1} \frac{P_i - P_0}{r_i^3} + \sum_{j=0}^{n-1} \frac{Q_j - P_0}{s_j^3} = -\mu P_0 \qquad \text{with } \mu > 0, \tag{3.1}$$

where

$$r_i = |P_i - P_0| = 2 \sin\left(\frac{\pi i}{n}\right),$$

$$s_j = |Q_j - P_0| = 2\left(\sin^2\left(\frac{\pi j}{n} + \frac{\beta}{2}\right) + a^2\right)^{\frac{1}{2}}.$$

Since $P_0 = (1, 0, a)$, the inner products $(0,1,0) \cdot \mu P_0$ and $(a, 0, -1) \cdot \mu P_0$ are zero, and from (3.1) we have

$$
\begin{aligned}
F(\beta, a) &:= \sum_{j=0}^{n-1} \frac{\sin\left(\frac{2\pi j}{n} + \beta\right)}{\left(\sin^2\left(\frac{\pi j}{n} + \frac{\beta}{2}\right) + a^2\right)^{\frac{3}{2}}} = 0, \\
G(\beta, a) &:= \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)} - \sum_{j=0}^{n-1} \frac{\cos^2\left(\frac{\pi j}{n} + \frac{\beta}{2}\right)}{\left(\sin^2\left(\frac{\pi j}{n} + \frac{\beta}{2}\right) + a^2\right)^{\frac{3}{2}}} = 0.
\end{aligned} \tag{3.2}
$$

Notice that the equations (3.2) imply that the vectors $\sum_{i=1}^{n-1} (P_i - P_0)/r_i^3 + \sum_{j=0}^{n-1} (Q_j - P_0)/s_j^3$ and P_0 are linearly dependent. Since the polyhedron of vertices $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1}$ is convex, the fisrt vector must be of the form λP_0 with $\lambda < 0$. Hence, the system (3.2) is equivalent to the equation (3.1).

Remark 3.1. It is easy to see that $F(0, a)$ and $F(\pi/n, a)$ are identically zero for all $a > 0$.

LEMMA 3.2. *For $\beta = 0$ and $n \geq 3$, there is a unique prism whose vertices $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1}$ form a symmetric central configuration.*

Proof. We write $G(0, a) = C - g(a)$ where $C = \sum_{j=1}^{n-1} (\sin(\pi j/n))^{-1} > 0$ and

$$
g(a) = \frac{1}{a^3} + \sum_{j=1}^{n-1} \frac{\cos^2\left(\frac{\pi j}{n}\right)}{\left(\sin^2\left(\frac{\pi j}{n}\right) + a^2\right)^{\frac{3}{2}}}.
$$

Note that $\lim_{a \to 0^+} g(a) = +\infty$ and that $\lim_{a \to +\infty} g(a) = 0$. Since g is continuous, there exists $a_0 \in (0, +\infty)$, such that $g(a_0) = C$. Moreover, since g is clearly decreasing it follows that such a_0 is unique. Hence, by Remark 3.1, the lemma follows. ∎

In the next lemma we think in the tetrahedron as a degenerate antiprism with $n = 2$.

LEMMA 3.3. *For $\beta = \pi/n$ and $n \geq 2$, there is a unique antiprism whose vertices $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1}$ form a symmetric central configuration.*

Proof. We define C as in the proof of Lemma 3.2. Then $G(\pi/n, a) = C - g(a)$, where

$$
g(a) = \sum_{j=0}^{n-1} \frac{\cos^2\left(\frac{\pi j}{n} + \frac{\pi}{2n}\right)}{\left(\sin^2\left(\frac{\pi j}{n} + \frac{\pi}{2n}\right) + a^2\right)^{\frac{3}{2}}}.
$$

We claim that $C < g(0)$. That is, we must prove that

$$\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi 2j}{2n}\right)} < \sum_{j=0}^{n-1} \frac{1-\sin^2\left(\frac{\pi(2j+1)}{2n}\right)}{\sin^3\left(\frac{\pi(2j+1)}{2n}\right)},$$

or equivalently

$$\sum_{j=1}^{2n-1} \sin^{-1}\left(\frac{\pi j}{2n}\right) < \sum_{j=0}^{n-1} \sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right).$$

Since $\sum_{j=1}^{2n-1} \sin^{-1}(\pi j/2n) = 1 + 2\sum_{j=1}^{n-1} \sin^{-1}(\pi j/2n)$, the proof of the claim is reduced to show that

$$A_n := \sum_{j=0}^{n-1} \left(\sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right) - 2\sin^{-1}\left(\frac{\pi(j+1)}{2n}\right)\right) \geq 0.$$

It is easy to see that $A_2 = 2(\sqrt{2}-1)$ and $A_3 = 11 - 4\sqrt{3}/3$. Now, we shall prove that $A_n > 0$ for $n \geq 4$. Clearly

$$\begin{aligned}
A_n &= \sum_{j=0}^{n-1} \sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right) - 2\sum_{j=0}^{[(n-1)/2]} \sin^{-1}\left(\frac{\pi(2j+1)}{2n}\right) \\
&\quad -2\sum_{j=1}^{[n/2]} \sin^{-1}\left(\frac{\pi 2j}{2n}\right) \\
&= \sum_{j=0}^{[(n-2)/4]} \sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right) + \sum_{j=[(n-2)/4]+1}^{-[-3n/4]-1} \sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right) \\
&\quad + \sum_{j=-[-3n/4]}^{n-1} \sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right) - 2\sum_{j=0}^{[(n-2)/4]} \sin^{-1}\left(\frac{\pi(2j+1)}{2n}\right) \qquad (3.3) \\
&\quad -2\sum_{j=[(n-2)/4]+1}^{[(n-1)/2]} \sin^{-1}\left(\frac{\pi(2j+1)}{2n}\right) \\
&\quad -2\sum_{j=1}^{[n/4]} \sin^{-1}\left(\frac{\pi 2j}{2n}\right) - 2\sum_{j=[n/4]+1}^{[n/2]} \sin^{-1}\left(\frac{\pi 2j}{2n}\right) \\
&= B_n + C_n + D_n + E_n,
\end{aligned}$$

where

$$B_n := \sum_{j=0}^{[(n-2)/4]} \left(\sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right) - 2\sin^{-1}\left(\frac{\pi(2j+1)}{2n}\right)\right),$$

$$C_n := \sum_{j=1}^{[n/4]} \left(\sin^{-3}\left(\frac{\pi(2j-1)}{2n}\right) - 2\sin^{-1}\left(\frac{\pi 2j}{2n}\right)\right),$$

$$D_n := \sum_{j=[(n-2)/4]+1}^{-[-3n/4]-1} \sin^{-3}\left(\frac{\pi(2j+1)}{2n}\right),$$

$$E_n := -2 \sum_{j=[(n-2)/4]+1}^{[(n-1)/2]} \sin^{-1}\left(\frac{\pi(2j+1)}{2n}\right)$$

$$-2 \sum_{j=[n/4]+1}^{[n/2]} \sin^{-1}\left(\frac{\pi 2j}{2n}\right),$$

and $[a]$ denotes the integer part function of $a \in \mathbb{R}$.

Since $\sin x - 2\sin^3 x \geq 0$ for $x \in [0, \pi/4]$, we have that

$$B_n \geq \sin^{-3}\left(\frac{\pi}{2n}\right) - 2\sin^{-1}\left(\frac{\pi}{2n}\right).$$

On the other hand,

$$\sin^{-3}\left(\frac{\pi}{2n}\right) > \left(\frac{\pi}{2n}\right)^{-3} = \frac{8n^3}{\pi^3},$$

because $\sin x < x$ for all $x > 0$; and

$$-\sin^{-1}\left(\frac{\pi}{2n}\right) > -\left(\frac{\pi}{2n} - \frac{\pi^3}{48n^3}\right)^{-1} = -\frac{48n^3}{24\pi n^2 - \pi^3},$$

because $x - x^3/6 < \sin x$ for all $x > 0$. Therefore,

$$B_n \geq \frac{8n^3}{\pi^3} - \frac{96n^3}{24\pi n^2 - \pi^3} > \frac{8n^3}{\pi^3} - \frac{96n^3}{23\pi n^2} > \frac{8n^3}{\pi^3} - 2n, \tag{3.4}$$

because $n \geq 4$.

From the definition of C_n it follows immediately that

$$C_n \geq \sum_{j=1}^{[n/4]} \left(\sin^{-3}\left(\frac{\pi j}{n}\right) - 2\sin^{-1}\left(\frac{\pi j}{n}\right)\right) \geq 0, \tag{3.5}$$

because $\sin x - 2\sin^3 x \geq 0$ for $x \in [0, \pi/4]$.

Since $\sin^{-3} x \geq 1$ for $x \in [\pi/4, 3\pi/4]$, and the number of summands in D_n is $-[-3n/4] - [(n-2)/4] - 1 \geq n/2$, it follows that

$$D_n \geq \frac{n}{2}. \tag{3.6}$$

Since $\sin^{-1} x \leq \sqrt{2}$ for all $x \in [\pi/4, 3\pi/4]$, and the number of summands in E_n is $-[-3n/4] - [(n-2)/4] - 1 \leq (n+1)/2$, it follows that

$$E_n \geq -2 \cdot \frac{n+1}{2} \cdot \sqrt{2} = -(n+1)\sqrt{2}. \tag{3.7}$$

In short, from (3.3) to (3.7) we obtain that

$$\begin{aligned} A_n &\geq \frac{8}{\pi^3} n^3 - 2n + \frac{n}{2} - (n+1)\sqrt{2} \\ &= \frac{8}{\pi^3} n^3 - \left(\frac{3}{2} + \sqrt{2}\right) n - \sqrt{2} > 0, \end{aligned}$$

for all $n \geq 4$. Hence, the claim is proved.

Since $\lim_{a \to +\infty} g(a) = 0$, $C < g(0)$ and g is continuous, there exists $a_0 > 0$ such that $g(a_0) = C$, or equivalently $G(\pi/n, a_0) = 0$. A such a_0 is unique because g is decreasing in $(0, +\infty)$. Now, by Remark 3.1, the lemma is proved. ∎

Proof of Corollary B. The proof follows immediately from Lemmas 3.2 and 3.3. ∎

From §2, the $4q$ vertices of any prism whose faces consist of two non-regular $2q$-gons connected by $2q$ rectangles which are the orbit of one of them by the action of $[2, q]$ (modulo rotations and homotheties) are $(\cos(2\pi i/q), \sin(2\pi i/q), \pm a)$ and $(\cos((2\pi i/q) + \beta), \sin((2\pi i/q) + \beta), \pm a)$, with $i = 0, \dots, q-1$, $\beta \in (0, \pi/q)$ and $a \neq 0$. Then, by using Lemma 1.1, we obtain numerically that the vertices of such prisms cannot form a symmetric central configuration. Hence, by Lemma 3.2, *the unique prisms which are symmetric central configurations have regular bases.* Table II gives the value a_0 (see Lemma 3.2) for such symmetric central configurations when the bases of the prism are regular q-gons for $q = 3, 4, \dots, 20$.

The $2q$ vertices of any antiprism are $(\cos(2\pi i/q), \sin(2\pi i/q), a)$ and $(\cos((2\pi i/q) + \beta), \sin((2\pi i/q) + \beta), -a)$, with $\beta \in (0, \pi/q)$, $a \neq 0$ and $i = 0, \dots, q-1$. By using the equations (3.2), it follows numerically that *if $\beta \neq \pi/q$ then such vertices cannot form a symmetric central configuration.* Therefore, by Lemma 3.3, the unique antiprisms which are symmetric central configurations are given in Table III for $q = 3, 4, \dots, 20$; where a_0 is defined in the proof of the Lemma 3.3.

Table II. Prisms

q	a_0		q	a_0
3	0.7935817272		12	0.5498498147
4	0.7071067812	(cube)	13	0.5427483527
5	0.6596217680		14	0.5364294665
6	0.6292858889		15	0.5307529421
7	0.6077667978		16	0.5256121443
8	0.5913463399		17	0.5209241302
9	0.5781738270		18	0.5166231776
10	0.5672349011		19	0.5126563880
11	0.5579234481		20	0.5089806068

The $4q$ vertices of any pseudo-prism are $(\cos((\pi i/q)+\beta), \sin((\pi i/q)+\beta), a)$, $(\cos(\pi i/q), \sin(\pi i/q), a)$, $(\cos((\pi j/q)+\beta), \sin((\pi j/q)+\beta), -a)$ and $(\cos(\pi j/q)$, $\sin(\pi j/q), -a)$ with $\beta \in (0, \pi/q)$, $a \neq 0$, $i = 0, 2, 4, \ldots, 2q-2$ and $j = 1, 3, 5, \ldots, 2q-1$. By using Lemma 1.1 we obtain numerically that *the vertices of such pseudo-prisms cannot form a symmetric central configuration.*

LEMMA 3.4. *The cube is the unique regular prism whose vertices form a symmetric central configuration.*

Proof. By using the notation of Lemma 3.2, a prism will be regular if $\beta = 0$ and $a = \sin(\pi/n)$. A such prism will be a symmetric central configuration if $G(0, a) = 0$. Since for $n \geq 9$ we have that

Table III. Antiprisms

q	a_0		q	a_0
3	0.7071067812	(octahedron)	12	0.5498299975
4	0.6766948174		13	0.5427400741
5	0.6482075959		14	0.5364259884
6	0.6248646668		15	0.5307514733
7	0.6060207350		16	0.5256115211
8	0.5906469904		17	0.5209238646
9	0.5778905566		18	0.5166230640
10	0.5671190812		19	0.5126563392
11	0.5578757091		20	0.5089805858

Table IV. Regular prisms and antiprisms

n	$G(0,\sin(\pi/n))$	$G(\pi/n,(\sin^2(\pi/n)-\sin^2(\pi/2n))^{1/2})$
3	0.50	0 (octahedron)
4	0 (cube)	−1.22
5	−1.83	−4.21
6	−5.43	−9.50
7	−11.22	−17.65
8	−19.61	−29.18
9	−31.01	−44.60
10	−45.84	−64.43
11	−64.48	−89.19

$$
\begin{aligned}
G(0,a) &\le \sum_{j=1}^{n-1} \sin^{-1}\left(\frac{\pi j}{n}\right) - \sin^{-3}\left(\frac{\pi}{n}\right) \\
&\le (n-1)\sin^{-1}\left(\frac{\pi}{n}\right) - \sin^{-3}\left(\frac{\pi}{n}\right) \\
&= \frac{(n-1)\sin^2\left(\frac{\pi}{n}\right)-1}{\sin^3\left(\frac{\pi}{n}\right)} \le \frac{(n-1)\pi^2-n^2}{n^2\sin^3\left(\frac{\pi}{n}\right)} < 0,
\end{aligned}
$$

it follows that the regular prism with $2n$ vertices and $n \ge 9$ cannot be a symmetric central configuration. From Table IV, the lemma follows. ∎

LEMMA 3.5. *The octahedron is the unique regular antiprism whose vertices form a symmetric central configuration.*

Proof. By using the notation of Lemma 3.3 an antiprism will be regular if $\beta = \pi/n$ and $a = (\sin^2(\pi/n) - \sin^2(\pi/2n))^{1/2}$. A such antiprism will be a symmetric central configuration if $G(\pi/n, a) = 0$. Since for $n \ge 12$ we have that

$$
\begin{aligned}
G\left(\frac{\pi}{n},a\right) &\le \sum_{j=1}^{n-1} \sin^{-1}\left(\frac{\pi j}{n}\right) - \cos^2\left(\frac{\pi}{2n}\right)\sin^{-3}\left(\frac{\pi}{n}\right) \\
&\le (n-1)\sin^{-1}\left(\frac{\pi}{n}\right) - \left(1-\sin^2\left(\frac{\pi}{2n}\right)\right)\sin^{-3}\left(\frac{\pi}{n}\right) \\
&= \left((n-1)\sin^2\left(\frac{\pi}{n}\right) + \sin^2\left(\frac{\pi}{2n}\right) - 1\right)\sin^{-3}\left(\frac{\pi}{n}\right) \\
&< \left((n-1)\frac{\pi^2}{n^2} + \frac{\pi}{2n} - 1\right)\sin^{-3}\left(\frac{\pi}{n}\right) \\
&< \left((n-1)\frac{10}{n^2} + \frac{2}{n} - 1\right)\sin^{-3}\left(\frac{\pi}{n}\right) = \frac{12n-10-n^2}{n^2\sin^3\left(\frac{\pi}{n}\right)} < 0,
\end{aligned}
$$

it follows that the regular antiprism with $2n$ vertices and $n \geq 12$ cannot be a symmetric central configuration. From Table IV, the lemma follows. ∎

4. ORBITS BY THE ACTION OF $[3,3]$ AND $[3,3]^+$

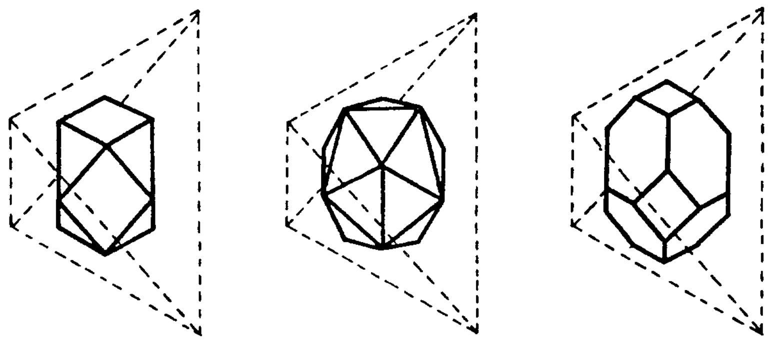
In §2 we have seen that the orbits of a point of S^2 by the action of $[3,3]$ or $[3,3]^+$ are the vertices of a regular tetrahedron, a regular octahedron, a truncated regular tetrahedron, a parallelly bevelled regular tetrahedron, a non-parallelly bevelled regular tetrahedron or a parallelly bevelled truncated regular tetrahedron.

It is well-known that *the regular tetrahedron, the regular octahedron, the cuboctahedron and the regular icosahedron are symmetric central configurations* [3].

The cuboctahedron is a special parallelly bevelled regular tetrahedron (see Figure 5(a)). By using Lemma 1.1, we obtain numerically that this *is the unique parallelly bevelled regular tetrahedron (modulo rotations and homotheties) which is a symmetric central configuration.*

The regular icosahedron is a special non-parallelly bevelled regular tetrahedron (see Figure 5(b)). By using Lemma 1.1, we obtain numerically that this *is the unique non-parallelly bevelled regular tetrahedron which is a symmetric central configuration.*

A truncated regular octahedron is a special parallelly bevelled truncated regular tetrahedron (see Figure 5(c)). By using Lemma 1.1, we obtain numerically that *there exists a unique parallelly bevelled truncated regular tetrahedron which is a symmetric central configuration* and, in fact, this polyhedron is a truncated regular octahedron. We shall show the existence of this symmetric central configuration in the next section.



(a) Cuboctahedron (b) Icosahedron (c) Truncated octahedron

Figure 5.

LEMMA 4.1. *There exists a truncated regular tetrahedron whose vertices form a symmetric central configuration.*

Proof. Consider the following points:

$$P_1 = (3-2a, -a, a), \quad P_2 = (a, 2a-3, a),$$
$$P_3 = (a, -a, 3-2a), \quad P_4 = (a, 3-2a, -a),$$
$$P_5 = (3-2a, a, -a), \quad P_6 = (a, a, 2a-3),$$
$$P_7 = (2a-3, a, a), \quad P_8 = (-a, a, 3-2a),$$
$$P_9 = (-a, 3-2a, a), \quad P_{10} = (-a, 2a-3, -a),$$
$$P_{11} = (-a, -a, 2a-3), \quad P_{12} = (2a-3, -a, -a),$$

with $0 < a < 1$. These points are the vertices of a truncated regular tetrahedron. In order to study what of these truncated regular tetrahedra are symmetric central configurations, we must solve system (1.3) for $(q_1, \ldots, q_{2n}) = (P_1, \ldots, P_{12})$, namely

$$\sum_{i=2}^{12} \frac{P_i - P_1}{r_i^3} = -\mu P_1 \qquad \text{with } \mu > 0, \tag{4.1}$$

where $r_i = |P_i - P_1|$.

Since $P_1 = (3-2a, -a, a)$, the inner products $(0,1,1) \cdot \mu P_1$ and $(a, 3, 2a) \cdot \mu P_1$ are zero. Now it is easy to see that equation (4.1) is equivalent to the system

$$f(a) := \sum_{i=2}^{12} \frac{(0,1,1) \cdot P_i}{r_i^3} = 0,$$

$$g(a) := \sum_{i=2}^{12} \frac{(a,3,2a) \cdot P_i}{r_i^3} = 0.$$

It is easy to check that $f(a)$ is identically zero for all $a \in (0,1)$. Now,

$$g(a) = -\frac{1}{6\sqrt{2}(1-a)^2} + \frac{3a - 2a^2}{\sqrt{2}(7a^2 - 12a + 9)^{3/2}} + \frac{3-2a}{8\sqrt{2}a^2}$$
$$+ \frac{2a-3}{4\sqrt{a}(5a^2 - 12a + 9)^{3/2}} + \frac{4a^2 - 15a + 9}{2\sqrt{2}(3-a)^3} = 0.$$

Since $\lim_{a \to 0^+} g(a) = +\infty$, $\lim_{a \to 1^-} g(a) = -\infty$ and g is a continuous function, we have that there exists $a_0 \in (0,1)$ such that $g(a_0) = 0$. Thus, for such a_0, the points $P_1, \ldots, P_{12}$ form a symmetric central configuration and the lemma is proved. ■

By using Lemma 1.1, we obtain numerically that *there exists a unique truncated regular tetrahedron which is a symmetric central configuration.* Furthermore $g(a) = 0$ for $a = 0.5618684192 \pm 10^{-10}$ and this truncated regular tetrahedron is not regular, i.e. its edges are not congruent.

5. ORBITS BY THE ACTION OF $[3,4], [3^+,4]$ AND $[3,4]^+$

In §2 we have seen that the orbits of a point of S^2 by the action of $[3,4], [3^+,4]$ or $[3,4]^+$ are the vertices of a cube, a regular octahedron, a cuboctahedron, a truncated cube, a truncated regular octahedron, a parallelly bevelled cube, a snub cube or a parallelly bevelled truncated cube.

It is well-known that *the cube, the regular octahedron and the cuboctahedron are symmetric central configurations* [3].

Consider the following points:

$$\begin{aligned}
&P_1 = (1, a, b), \quad P_2 = (1, -b, a), \quad P_3 = (1, -a, -b),\\
&P_4 = (1, b, -a), \quad P_5 = (-a, 1, b), \quad P_6 = (b, 1, a),\\
&P_7 = (a, 1, -b), \quad P_8 = (-b, 1, -a), \quad P_9 = (-b, a, 1),\\
&P_{10} = (-a, -b, 1), \quad P_{11} = (b, -a, 1), \quad P_{12} = (a, b, 1),\\
&P_{13} = (-1, a, -b), \quad P_{14} = (-1, b, a), \quad P_{15} = (-1, -a, b),\\
&P_{16} = (-1, -b, -a), \quad P_{17} = (-a, -1, -b), \quad P_{18} = (-b, -1, a),\\
&P_{19} = (a, -1, b), \quad P_{20} = (b, -1, -a), \quad P_{21} = (b, a, -1),\\
&P_{22} = (-a, b, -1), \quad P_{23} = (-b, -a, -1), \quad P_{24} = (a, -b, -1),
\end{aligned}$$

with $0 \leq a, b \leq 1$. Let $r_i = |P_i - P_1|$ for $i = 2, 3, \ldots, 24$.

The set $\{P_1, \ldots, P_{24}\}$ is an orbit by the action of $[3,4]^+$. So, if the points P_i are all different, then $(P_1, \ldots, P_{24})$ is a symmetric central configuration if and only if it satisfies the equation

$$\sum_{i=2}^{24} \frac{P_i - P_1}{r_i^3} = -\mu P_1 \qquad \text{with } \mu > 0. \tag{5.1}$$

Since $P_1 = (1, a, b)$, the inner products $(-a, 1, 0) \cdot \mu P_1$ and $(-b, 0, 1) \cdot \mu P_1$ are zero. Now it is easy to see that equation (5.1) is equivalent to the system

$$\begin{aligned}
f(a, b) &:= \sum_{i=2}^{24} \frac{(-a, 1, 0) \cdot P_i}{r_i^3} = 0,\\
g(a, b) &:= \sum_{i=2}^{24} \frac{(-b, 0, 1) \cdot P_i}{r_i^3} = 0.
\end{aligned} \tag{5.2}$$

If $a = 1$ and $0 < b < 1$, then it is easy to see that the P_i are the vertices of a truncated cube, and $f(1, b)$ is identically zero for all $0 < b < 1$.

LEMMA 5.1. *There exists a truncated cube whose vertices form a symmetric central configuration.*

Proof. Let $h(b) = g(1, b)$ with $0 < b < 1$. Then

$$\begin{aligned} h(b) = & -\frac{b}{\sqrt{2}(1+b^2)^{3/2}} - \frac{b}{4(1+b^2)^{3/2}} + \frac{b}{4} + \frac{2+b}{2\sqrt{2}(1-b)^2} \\ & - \frac{1}{4b^2} + \frac{b-2}{2\sqrt{2}(1+b)^2} + \frac{b}{\sqrt{2}(3+b^2)^{3/2}} \\ & + \frac{2+b-b^2}{2\sqrt{2}(3-2b+b^2)^{3/2}} + \frac{b}{8\sqrt{2}} + \frac{-2+b+b^2}{2\sqrt{2}(3+2b+b^2)^{3/2}}. \end{aligned}$$

Since $\lim_{b\to 0^+} h(b) = -\infty$, $\lim_{b\to 1^-} h(b) = +\infty$ and h is a continuous function, we have that there exists $b_0 \in (0, 1)$ such that $h(b_0) = 0$. Thus for $a = 1$ and $b = b_0$, $(P_1, \ldots, P_{24})$ is a symmetric central configuration. ∎

By using Lemma 1.1, we obtain numerically that *there is a unique truncated cube which is a symmetric central configuration.* Furthermore $h(b) = 0$ for $b = 0.3699054980 \pm 10^{-10}$, and this polyhedron is not regular.

If $a = 0$ and $0 < b < 1$, then the P_i are the vertices of a truncated regular octahedron and it is easy to see that $f(0, b)$ is identically zero for all $0 < b < 1$. By using the same kind of arguments that in Lemma 5.1, we can show the following lemma.

LEMMA 5.2. *There exists a truncated regular octahedron whose vertices form a symmetric central configuration.* ∎

By using Lemma 1.1, we obtain numerically that *there is a unique truncated regular octahedron which is a symmetric central configuration.* Furthermore $g(0, b) = 0$ for $b = 0.5312723138 \pm 10^{-10}$, and this polyhedron is not regular.

If $a = b$ and $0 < a < 1$, then the P_i are the vertices of a parallelly bevelled cube and it is easy to see that $f(a, a) = g(a, a)$ for all $0 < a < 1$. The next lemma follows similarly to Lemma 5.1.

LEMMA 5.3. *There exists a parallelly bevelled cube whose vertices form a symmetric central configuration.* ∎

By using Lemma 1.1, we obtain numerically that *there is a unique parallelly bevelled cube which is a symmetric central configuration.* Furthermore $f(a,a) = 0$ for $a = 0.4037831820 \pm 10^{-10}$, and this polyhedron is not regular.

If $0 < a < b < 1$, then the P_i are the vertices of a snub cube.

LEMMA 5.4. *There exists a sunb cube whose vertices form a symmetric central configuration.*

Proof. The functions $f(a,b)$ and $g(a,b)$ defined in (5.2) are continuous in $T = \{(a,b) \mid 0 \le a \le b \le 1\} \setminus \{(0,0),(0,1),(1,1)\}$.

We know that $f(0,b) = 0$ for all $0 < b < 1$. An easy calculation show us that $\frac{\partial f}{\partial a}(0,b) < 0$ for all $0 < b < 1$. Consider the segments $S_b = \{(\lambda, \lambda + (1-\lambda)b) \mid 0 \le \lambda \le 1\}$ for each $b \in (0,1)$. Now it is easy to see that there exists $\varepsilon_b > 0$ such that

$$f(\lambda, \lambda + (1-\lambda)b) < 0 \quad \text{if} \quad 0 < \lambda < \varepsilon_b, \text{ and}$$
$$f(\lambda, \lambda + (1-\lambda)b) > 0 \quad \text{if} \quad 1 - \varepsilon_b < \lambda < 1,$$

because $\lim_{\lambda \to 1^-} f(\lambda, \lambda + (1-\lambda)b) = +\infty$. So there exists $\lambda \in (0,1)$ such that $f(\lambda, \lambda + (1-\lambda)b) = 0$. Let λ_b be the smallest positive real number such that $f(\lambda_b, \lambda_b + (1-\lambda_b)b) = 0$. Set $Z_f = \{(\lambda_b, \lambda_b + (1-\lambda_b)b) \mid 0 < b < 1\}$. It is clear that Z_f is a connected continuous curve such that decomposes the interior of T into two regions T_1, T_2 with $f(a,b) < 0$ for all $(a,b) \in T_1$.

On the other hand, we can see that $g(a,1) = 0$ for all $0 < a < 1$. An easy calculation show us that $\frac{\partial g}{\partial b}(a,1) < 0$ for all $0 < a < 1$. Consider the segments $R_a = \{(\lambda a, \lambda) \mid 0 \le \lambda \le 1\}$ for each $a \in (0,1)$. Since $\lim_{\lambda \to 0^+} g(\lambda a, \lambda) = -\infty$ and $\frac{\partial g}{\partial b}(a,1) < 0$, there exists $\varepsilon_a > 0$ such that

$$g(\lambda a, \lambda) < 0 \quad \text{if} \quad 0 < \lambda < \varepsilon_a, \text{ and}$$
$$g(\lambda a, \lambda) > 0 \quad \text{if} \quad 1 - \varepsilon_a < \lambda < 1.$$

So there exists $\lambda \in (0,1)$ such that $g(\lambda a, \lambda) = 0$. Let λ'_a be the greatest real number in $(0,1)$ such that $g(\lambda'_a a, \lambda'_a) = 0$. Set $Z_g = \{(\lambda'_a a, \lambda'_a) \mid 0 < a < 1\}$. Z_g is a connected continuous curve such that decomposes the interior of T into two regions T_3, T_4 with $g(a,b) > 0$ for all $(a,b) \in T_4$.

It is easy to show that $\lim_{\lambda \to 0^+} f(\lambda a, 1 - \lambda + \lambda a) = -\infty$ for all $a \in (0,1]$ and $\lim_{\lambda \to 0^+} g(\lambda a, 1 - \lambda + \lambda a) = +\infty$ for all $a \in [0,1)$, so $T_1 \cap T_4 \ne \emptyset$. An easy computation show us that $f(25/81, 40/81) > 0$ and $g(25/81, 40/81) < 0$. Hence $Z_f \cap Z_g \ne \emptyset$. Thus there exists (a_0, b_0) inside T such that $f(a_0, b_0) = g(a_0, b_0) = 0$, and the lemma follows. ∎

By using Lemma 1.1, we obtain numerically that *there are exactly two snub cubes which are symmetric central configurations*, one for $a = 0.2904681239 \pm 10^{-10}$ and $b = 0.5124883051 \pm 10^{-10}$, and the other for $a = 0.5124883051 \pm 10^{-10}$ and $b = 0.2904681239 \pm 10^{-10}$. These polyhedra are not regular.

Consider the following points:

$$\begin{gathered} Q_1 = (1,-a,b), \quad Q_2 = (1,b,a), \quad Q_3 = (1,a,-b), \\ Q_4 = (1,-b,-a), \quad Q_5 = (a,1,b), \quad Q_6 = (-b,1,a), \\ Q_7 = (-a,1,-b), \quad Q_8 = (b,1,-a), \quad Q_9 = (b,a,1), \\ Q_{10} = (a,-b,1), \quad Q_{11} = (-b,-a,1), \quad Q_{12} = (-a,b,1), \\ Q_{13} = (-1,-a,-b), \quad Q_{14} = (-1,-b,a), \quad Q_{15} = (-1,a,b), \\ Q_{16} = (-1,b,-a), \quad Q_{17} = (a,-1,-b), \quad Q_{18} = (b,-1,a), \\ Q_{19} = (-a,-1,b), \quad Q_{20} = (-b,-1,-a), \quad Q_{21} = (-b,a,-1), \\ Q_{22} = (a,b,-1), \quad Q_{23} = (b,-a,-1), \quad Q_{24} = (-a,-b,-1), \end{gathered}$$

with $0 < a < b < 1$. Let $s_i = |Q_i - P_1|$ for $i = 1, \ldots, 24$. It is easy to see that the P_i and the Q_i are the vertices of a parallelly bevelled truncated cube. $(P_1, \ldots, P_{24}, Q_1, \ldots, Q_{24})$ is a symmetric central configuration if and only if satisfies the system

$$\begin{aligned} F(a,b) &:= \sum_{i=2}^{24} \frac{(-a,1,0) \cdot P_i}{r_i^3} + \sum_{i=1}^{24} \frac{(-a,1,0) \cdot Q_i}{s_i^3} = 0, \\ G(a,b) &:= \sum_{i=2}^{24} \frac{(-b,0,1) \cdot P_i}{r_i^3} + \sum_{i=1}^{24} \frac{(-b,0,1) \cdot Q_i}{s_i^3} = 0. \end{aligned} \tag{5.3}$$

LEMMA 5.5. *There exists a parallelly bevelled truncated cube whose vertices form a symmetric central configuration.*

Proof. The functions $F(a,b)$ and $G(a,b)$ defined in (5.3) are continuous in $T = \{(a,b) \mid 0 < a < b < 1\}$. Since $(-b,0,1) \cdot Q_1 = 0$, the function $G(a,b)$ is continuous for $0 \leq a < b < 1$. An easy calculation show us $\lim_{a \to 0^+} F(a,b) = -\infty$ and $\lim_{a \to b^-} F(a,b) = +\infty$ for all $0 < b < 1$, $\lim_{b \to 1^-} G(a,b) = +\infty$ and $\lim_{b \to a^+} G(a,b) = -\infty$ for all $0 < a < 1$, and $\lim_{\lambda \to 0^+} G(\lambda a, \lambda) = -\infty$ for all $0 \leq a < 1$.

Let a_b be the smallest positive real number such that $F(a_b, b) = 0$. Let b_a be the smallest real number such that $b_a > a$ and $G(a, b_a) = 0$. Set $Z_F = \{(a_b, b) \mid$

$0 < b < 1\}$ and $Z_G = \{(a, b_a) \mid 0 < a < 1\}$. Since $F(1/3, 2/3) > 0$ and $G(1/3, 2/3) > 0$, it is easy to see that $Z_F \cap Z_G \neq \emptyset$. Thus there exists $(a_0, b_0) \in T$ such that $F(a_0, b_0) = G(a_0, b_0) = 0$, and the lemma follows. ■

By using Lemma 1.1, we obtain numerically that *there is a unique parallelly bevelled truncated cube which is a symmetric central configuration.* Furthermore, $G(a, b) = F(a, b) = 0$ for $a = 0.5785659437 \pm 10^{-10}$ and $b = 0.2510277455 \pm 10^{-10}$, and this polyhedron is not regular.

6. ORBITS BY THE ACTION OF $[3,5]$ AND $[3,5]^+$

In §2 we have seen that the orbits of a point of S^2 by the action of $[3,5]$ or $[3,5]^+$ are the vertices of a regular icosahedron, a regular dodecahedron, a icosidodecahedron, a truncated regular icosahedron, a truncated regular dodecahedron, a parallelly bevelled regular icosahedron, a snub dodecahedron or a parallelly bevelled truncated regular icosahedron.

It is well-known that *the regular icosahedron, the regular dodecahedron and the icosidodecahedron are symmetric central configurations* [3].

Let g be the positive number such that $g^2 = g + 1$. It is well-known that the points

$$(\pm 6, 0, \pm 6g), (0, \pm 6g, \pm 6), (\pm 6g, \pm 6, 0)$$

are the twelve vertices of a regular icosahedron. The barycentres of their faces are the points

$$(0, \pm 2g, \pm 2g^3), (\pm 2g^3, 0, \pm 2g),$$
$$(\pm 2g, \pm 2g^3, 0), (\pm 2g^2, \pm 2g^2, \pm 2g^2),$$

and the midpoints of their edges are the points

$$(0, 0, \pm 6g),\ (0, \pm 6g, 0),\ (\pm 6g, 0, 0),$$
$$(\pm 3, \pm 3g, \pm 3g^2),\ (\pm 3g^2, \pm 3, \pm 3g),\ (\pm 3g, \pm 3g^2, \pm 3).$$

Let $\alpha, \beta \in [0, 1]$ such that $0 \leq \alpha + \beta \leq 1$. It is easy to see that the orbit of $P_1 = \alpha(-2g^2, 2g^2, 2g^2) + \beta(-3g^2, 3, 3g) + (1 - \alpha - \beta)(-6, 0, 6g)$ by the action of $[3,5]$ is the set

$$
\begin{aligned}
O_{\alpha,\beta} =\{&(\pm(6-6\alpha-6\beta), \pm 2g\alpha, \pm(6g-2g^{-1}\alpha)),\\
&(\pm 2g\alpha, \pm(6g-2g^{-1}\alpha), (\pm(6-6\alpha-6\beta)),\\
&(\pm(6g-2g^{-1}\alpha), (\pm(6-6\alpha-6\beta), \pm 2g\alpha),\\
&(\pm(6+4g^{-1}\alpha+3g^{-1}\beta), \pm 3\beta, \pm(6g-4g\alpha-3g\beta)),\\
&(\pm 3\beta, \pm(6g-4g\alpha-3g\beta), \pm(6+4g^{-1}\alpha+3g^{-1}\beta)),\\
&(\pm(6g-4g\alpha-3g\beta), \pm(6+4g^{-1}\alpha+3g^{-1}\beta), \pm 3\beta),\\
&(\pm(6-2g^{-2}\alpha+3g^{-1}\beta), \pm(2g^2\alpha+3\beta), \pm(6g+2(1-2g)\alpha-3g\beta)),\\
&(\pm(2g^2\alpha+3\beta), \pm(6g+2(1-2g)\alpha-3g\beta), \pm(6-2g^{-2}\alpha+3g^{-1}\beta)),\\
&(\pm(6g+2(1-2g)\alpha-3g\beta), \pm(6-2g^{-2}\alpha+3g^{-1}\beta), \pm(2g^2\alpha+3\beta)),\\
&(\pm(6-6\alpha-3\beta), \pm(2g\alpha+3g\beta), \pm(6g-2g^{-1}\alpha-3g^{-1}\beta)),\\
&(\pm(2g\alpha+3g\beta), \pm(6g-2g^{-1}\alpha-3g^{-1}\beta), \pm(6-6\alpha-3\beta)),\\
&(\pm(6g-2g^{-1}\alpha-3g^{-1}\beta), \pm(6-6\alpha-3\beta), \pm(2g\alpha+3g\beta))\\
&(\pm(6-2g^{-2}\alpha-3\beta), \pm(2g^2\alpha+3g\beta), \pm(6g+2(1-2g)\alpha-3g^{-1}\beta)),\\
&(\pm(2g^2\alpha+3g\beta), \pm(6g+2(1-2g)\alpha-3g^{-1}\beta), \pm(6-2g^{-2}\alpha-3\beta))\\
&(\pm(6g+2(1-2g)\alpha-3g^{-1}\beta), \pm(6-2g^{-2}\alpha-3\beta), \pm(2g^2\alpha+3g\beta))\}.
\end{aligned}
$$

Let $r_P = |P - P_1|$ for $P \in O_{\alpha,\beta} \setminus \{P_1\}$. The points of $O_{\alpha,\beta}$ form a symmetric central configuration if and only if they satisfy the equation

$$\sum_{P\in O_{\alpha,\beta}\setminus\{P_1\}} \frac{P-P_1}{r_P^3} = -\mu P_1 \qquad \text{with} \quad \mu > 0. \tag{6.1}$$

Suppose that $\alpha + \beta \neq 0$. Let $Q_1 = (g, (2g\alpha + 3g^2\beta)/(2g^2\alpha + 3\beta), 1)$ and $Q_2 = (3 - 4\alpha - 3\beta, -6 - 3g + 4\alpha + 3\beta, -3 + 3g + 4\alpha + 3\beta)$. Since the inner products $P_1 \cdot Q_1$ and $P_1 \cdot Q_2$ are zero, it is easy to see that the equation (6.1) is equivalent to the system

$$
\begin{aligned}
f_1(\alpha,\beta) &:= \sum_{P\in O_{\alpha,\beta}\setminus\{P_1\}} \frac{P\cdot Q_1}{r_P^3} = 0,\\
g_1(\alpha,\beta) &:= \sum_{P\in O_{\alpha,\beta}\setminus\{P_1\}} \frac{P\cdot Q_2}{r_P^3} = 0.
\end{aligned} \tag{6.2}
$$

If $\alpha = 0$ and $0 < \beta < 1$, then the elements of $O_{\alpha,\beta}$ are the vertices of a truncated regular icosahedron. Now $Q_1 = (g, g^2, 1)$. Since the inner products $(g, g^2, 1) \cdot (-3g^2, 3, 3g)$ and $(g, g^2, 1) \cdot (-6, 0, 6g)$ are zero, it is easy to see that $f_1(0, \beta)$ is identically zero for all $0 < \beta < 1$.

LEMMA 6.1. *There exists a truncated regular icosahedron whose vertices form a symmetric central configuration.*

Proof. An easy calculation give us that $\lim_{\beta\to 0^+} g_1(0,\beta) = +\infty$ and $\lim_{\beta\to 1^-} g_1(0,\beta) = -\infty$. Since $g_1(0,\beta)$ is a continuous function, the lemma follows. ∎

By using Lemma 1.1, we obtain numerically that *there is a unique truncated regular icosahedron which is a symmetric central configuration.* Furthermore $g_1(0,\beta) = 0$ for $\beta = 0.6806072959 \pm 10^{-10}$. This polyhedron is not regular.

If $\alpha = 1-\beta$ and $0 < \beta < 1$, then the elements of $O_{\alpha,\beta}$ are the vertices of a truncated regular dodecahedron. Let $P_1' = (-6+6\alpha+6\beta, -2g\alpha, 6g-2g^{-1}\alpha)$, $Q_1' = (1,0,0)$ and $Q_2' = (0, 3\alpha^{-1} - g^{-2}, 1)$. Since $P_1' \in O_{\alpha,\beta}$ and for $\alpha = 1-\beta$ we have that the inner products $P_1' \cdot Q_1'$ and $P_1' \cdot Q_2'$ are zero, it is easy to see that the points of $O_{1-\beta,\beta}$ form a symmetric central configuration if and only if they satisfy the system

$$f_2(\beta) := \sum_{P\in O_{1-\beta,\beta}\setminus\{P_1'\}} \frac{P\cdot Q_1'}{s_P^3} = 0,$$

$$g_2(\beta) := \sum_{P\in O_{1-\beta,\beta}\setminus\{P_1'\}} \frac{P\cdot Q_2'}{s_P^3} = 0$$

where $s_P = |P - P_1'|$ for $P \in O_{1-\beta,\beta} \setminus \{P_1'\}$. Since the inner products $Q_1' \cdot (0,-2g,2g^3)$ and $Q_1' \cdot (0,0,6g)$ are zero, it is easy to see that $f_2(\beta)$ is identicaly zero for all $0 < \beta < 1$.

LEMMA 6.2. *There exists a truncated regular dodecahedron whose vertices form a symmetric central configuration.*

Proof. An easy calculation give us that $\lim_{\beta\to 0^+} g_2(\beta) = -\infty$ and $\lim_{\beta\to 1^-} g_2(\beta) = +\infty$. Since $g_2(\beta)$ is a continuous function, the lemma follows. ∎

By using Lemma 1.1, we obtain numerically that *there is a unique truncated regular dodecahedron which is a symmetric central configuration.* Furthermore $g_2(\beta) = 0$ for $\beta = 0.5689512783 \pm 10^{-10}$. This polyhedron is not regular.

If $\beta = 0$ and $0 < \alpha < 1$, then the elements of $O_{\alpha,\beta}$ are the vertices of a parallelly bevelled regular icosahedron. Now $Q_1 = (g, g^{-1}, 1)$. Since the inner products $(g,g^{-1},1)\cdot(-2g^2,2g^2,2g^2)$ and $(g,g^{-1},1)\cdot(-6,0,6g)$ are zero, it is easy to see that $f_1(\alpha,0)$ is identically zero for all $0 < \alpha < 1$.

LEMMA 6.3. *There is a parallelly bevelled regular icosahedron whose vertices form a symmetric central configuration.*

Proof. An easy calculation give us that $\lim_{\alpha\to 0^+} g_1(\alpha,0) = +\infty$ and $\lim_{\alpha\to 1^-} g_1(\alpha,0) = -\infty$. Since $g_1(\alpha,0)$ is a continuous function, the lemma follows. ∎

By using Lemma 1.1, we obtain numerically that *there is a unique parallelly bevelled regular icosahedron which is a symmetric central configuration.* Furthermore, $g_1(\alpha,0) = 0$ for $\alpha = 0.6251797961 \pm 10^{-10}$. This polyhedron is not regular.

If $0 < \alpha, \beta, \alpha+\beta < 1$, then the elements of $O_{\alpha,\beta}$ are the vertices of a parallelly bevelled truncated regular icosahedron.

LEMMA 6.4. *There is a parallelly bevelled truncated regular icosahedron whose vertices form a symmetric central configuration.*

Proof. Let $P_2 = (-6+2g^{-2}\alpha+3\beta, 2g^2\alpha+3g\beta, 6g+2(1-2g)\alpha-3g^{-1}\beta)$. Since the inner products $P_2 \cdot Q_2$ is zero, we may define a continuous function

$$G(\alpha,\beta) = \begin{cases} g_1(\alpha,\beta) & \text{if} \quad 0<\alpha,\beta,\alpha+\beta<1, \\ 2g_1(\alpha,0) & \text{if} \quad 0<\alpha<1 \quad \text{and} \quad \beta=0. \end{cases}$$

An easy calculation give us that $\lim_{\alpha\to 0^+} G(\alpha,\lambda(1-\alpha)) = +\infty$ and $\lim_{\alpha\to 1^-} G(\alpha,\lambda(1-\alpha)) = -\infty$ for all $0 \le \lambda < 1$, and $\lim_{\beta\to 0^+} f_1(\lambda(1-\beta),\beta) = +\infty$ and $\lim_{\beta\to 1^-} f_1(\lambda(1-\beta),\beta) = -\infty$ for all $0<\lambda<1$.

Let a_λ be the smallest positive real number such that $G(a_\lambda,\lambda(1-a_\lambda)) = 0$. Let b_λ be the smallest positive real number such that $f_1(\lambda(1-b_\lambda),b_\lambda) = 0$. Set $Z_G = \{(a_\lambda,\lambda(1-a_\lambda)) \mid 0<\lambda<1\}$ and $Z_f = \{(\lambda(1-b_\lambda),b_\lambda) \mid 0<\lambda<1\}$. Since $f_1(15/41,20/41) < 0$ and $g_1(15/41,20/41) < 0$, it is easy to see that $Z_G \cap Z_f \neq \emptyset$. Thus there exists (a_0,b_0) such that $0 < a_0, b_0, a_0+b_0 < 1$ and $f_1(a_0,b_0) = g_1(a_0,b_0) = 0$ and the lemma follows. ∎

By using Lemma 1.1, we obtain numerically that *there is a unique parallelly bevelled truncated regular icosahedron which is a symmetric central configuration.* Furthermore, $f_1(\alpha,\beta) = g_1(\alpha,\beta) = 0$ for $\alpha = 0.3639173493 \pm 10^{-10}$ and $\beta = 0.3709253130 \pm 10^{-10}$. This polyhedron is not regular.

Let $O'_{\alpha,\beta}$ be the orbit of P_1 by the action of $[3,5]^+$. Let $Q''_2 = (-3-6g+4g\alpha+2\alpha+3g\beta+3\beta, 6+3g-4\alpha-3\beta, -3g^2-2g\alpha-3g\beta)$. Since Q_1, Q''_2 are linearly independent for all $\alpha,\beta \in T \setminus \{(0,0),(1,0),(0,1)\}$, where $T = \{(x,y) \in \mathbf{R}^2 \mid 0 \le x,y,x+y \le 1\}$ and the inner products $P_1 \cdot Q_1$ and $P_1 \cdot Q''_2$ are zero, it is easy to see that the points of $O'_{\alpha,\beta}$ form a symmetric central configuration if and only if they satisfy the system

$$\bar{f}(\alpha,\beta) := \sum_{P\in O'_{\alpha,\beta}\setminus\{P_1\}} \frac{P\cdot Q_1}{r_P^3} = 0,$$

$$\bar{g}(\alpha,\beta) := \sum_{P\in O'_{\alpha,\beta}\setminus\{P_1\}} \frac{P\cdot Q''_2}{r_P^3} = 0.$$

If $0 < \alpha,\beta,\alpha+\beta < 1$, then the elements of $O'_{\alpha,\beta}$ are the vertices of a snub dodecahedron.

LEMMA 6.5. *There exists a snub dodecahedron whose vertices form a symmetric central configuration.*

Proof (sketch). We see that $\frac{\partial\bar{f}}{\partial\alpha}(0,\beta) < 0$ for all $0 < \beta < 1$. Since $\bar{f}(0,\beta) = 0$ for all $0 < \beta < 1$, for each β there exists $\varepsilon_\beta > 0$ such that $\bar{f}(\alpha,\beta) < 0$ for all $0 < \alpha < \varepsilon_\beta$. We see that there exists $\varepsilon > 0$ such that $\frac{\partial\bar{f}}{\partial\beta}(\alpha,0) > 0$ for all $1-\varepsilon < \alpha < 1$. Since $\lim_{\lambda\to 1^-}\bar{f}(1-\lambda,\lambda) = +\infty$, for each $\beta\in(0,1)$ there exists $\lambda\in(0,1)$ such that $\bar{f}(1-\lambda,\beta\lambda) = 0$. Let λ_β be the smallest positive number such that $\bar{f}(1-\lambda_\beta,\beta\lambda_\beta) = 0$. Set $Z_f = \{(1-\lambda_\beta,\beta\lambda_\beta) \mid 0 < \beta < 1\}$. It is clear that Z_f is a connected continuous curve.

We see that $\frac{\partial\bar{g}}{\partial\beta}(\alpha,1-\alpha) < 0$ for all $0 < \alpha < 1$, $\lim_{\lambda\to 1^-}\bar{g}(0,\lambda) = +\infty$ and $\lim_{\lambda\to 1^-}\bar{g}(\lambda,0) = +\infty$. So there exists $\varepsilon > 0$ such that $\bar{g}(\lambda\alpha,\lambda(1-\alpha)) > 0$ for all $1-\varepsilon < \lambda < 1$ and for all $0 \le \alpha \le 1$. We see that $\lim_{\lambda\to 0^+}\bar{g}(\lambda\alpha,\lambda(1-\alpha)) = +\infty$ for all $0 \le \alpha \le 1$. Thus for each $\alpha\in[0,1]$ there exists $\lambda\in(0,1)$ such that $\bar{g}(\lambda\alpha,\lambda(1-\alpha)) = 0$. Let λ'_α be the greatest real number in $(0,1)$ such that $\bar{g}(\lambda'_\alpha\alpha,\lambda'_\alpha(1-\alpha)) = 0$. Set $Z_g = \{(\lambda'_\alpha\alpha,\lambda'_\alpha(1-\alpha)) \mid 0 < \alpha < 1\}$. It is clear that Z_g is a connected continuous curve. Since $\bar{f}(21/41,5/41) > 0$ and $\bar{g}(21/41,5/41) < 0$, it is easy to see that $Z_f \cap Z_g \neq \emptyset$. Thus the lemma follows. ■

By using Lemma 1.1, we obtain numerically that *there are exactly two snub dodecahedra which are symmetric central configurations.* The sets of vertices of these polyhedra are $O'_{\alpha,\beta}$ and $O_{\alpha,\beta}\setminus O'_{\alpha,\beta}$ for $\alpha = 0.4514835820 \pm 10^{-10}$ and $\beta = 0.2278940667 \pm 10^{-10}$. These polyhedra are not regular.

Proof of Theorem A. The existence of the symmetric central configurations given in Table I follows by Lemmas 3.2, 3.3, 4.1, 5.1, 5.2, 5.3, 5.4, 5.5, 6.1, 6.2, 6.3, 6.4 and 6.5. The uniqueness has been obtained numerically. ■

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